Numerical Analysis of Unsymmetrical Bending of Shells of Revolution

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A general numerical procedure, based on the linear theory of Sanders, is developed for the elastic stress and deflection analysis of a shell of revolution subjected to arbitrary loads and temperatures. The shell may have variable and discontinuous, but axisymmetric, geometrical and mechanical properties. The procedure involves the expansion of all pertinent load, stress, and deformation variables into Fourier series in the circumferential direction; the individual Fourier components of stress and deflection then are found separately by matrix solution of the finite-difference forms of appropriate differential equations in the meridional coordinate.

Nomenclature

a = reference length
$a_1, a_2, \ldots a_{36}$ = coefficients defined in Appendix A
b = nondimensional membrane stiffness [Eq. (33)]
c_1, c_2, c_3, c_4 = elements of column matrix e, defined in Appendix A
d = nondimensional bending stiffness [Eq. (34)]
$e_\xi, e_\theta, e_{\xi\theta}$ = Fourier coefficients for membrane strains [Eq. (23)]
$\hat{f}_\xi$ = Fourier coefficient for effective transverse shear [Eq. (45)]
h_0 = reference thickness
$k_\xi, k_\theta, k_{\xi\theta}$ = Fourier coefficients for bending distortion [Eq. (24)]
$m_\xi, m_\theta, m_{\xi\theta}$ = Fourier coefficients for bending moments [Eq. (20)]
p, p_ξ, p_θ = Fourier coefficients for loads [Eq. (25)]
q, q_ξ, q_θ = shell loads per unit area (Fig. 2d)
r = normal distance from shell to axis (Fig. 1)
s = meridional shell coordinate (Fig. 1)
$t_\xi, t_\theta, t_{\xi\theta}$ = Fourier coefficients for membrane forces [Eq. (19)]
$\hat{t}_{\xi\theta}$ = Fourier coefficient for effective membrane shear [Eq. (45)]
u_ξ, u_θ = Fourier coefficients for meridional and circumferential displacements [Eq. (21)]
w = Fourier coefficient for normal displacement [Eq. (21)]
E = Young's modulus
$M_\xi, M_\theta, M_{\xi\theta}$ = bending moments per unit length (Fig. 2c)
$\bar{M}_{\xi\theta}$ = modified twisting moment [Eq. (9)]
$N_\xi, N_\theta, N_{\xi\theta}$ = membrane forces per unit length (Fig. 2a)
$\bar{N}_{\xi\theta}$ = modified membrane shear [Eq. (8)]
$\hat{N}_{\xi\theta}$ = effective (boundary) membrane shear [Eq. (43)]
Q_ξ, Q_θ = transverse forces per unit length (Fig. 2b)
$\hat{Q}_\xi$ = effective (boundary) transverse shear [Eq. (44)]
R_ξ, R_θ = radii of curvature [Fig. 1; Eq. (2)]
T = temperature change
U_ξ, U_θ = meridional and circumferential displacements (Fig. 3a)
W = normal displacement (Fig. 3a)
α = thermal expansion coefficient
γ = ρ'/ρ
$\epsilon_\xi, \epsilon_\theta, \epsilon_{\xi\theta}$ = membrane strains [Eq. (12)]
θ = circumferential angle (Fig. 1)
$\kappa_\xi, \kappa_\theta, \kappa_{\xi\theta}$ = bending distortions [Eq. (13)]
λ = h_0/a
ν = Poisson's ratio
ξ = nondimensional meridional coordinates (s/a)
ρ = r/a
$\sigma_\xi, \sigma_\theta, \sigma_{\xi\theta}$ = meridional, circumferential, and shear stresses
$\varphi_\xi, \varphi_\theta$ = Fourier coefficients for rotation
ψ = inclination change at discontinuity (Fig. 5)
$\omega_\theta, \omega_\xi$ = nondimensional curvatures [Eqs. (3) and (4)]
Δ = interval size (in units of ξ) between stations
Φ_ξ, Φ_θ = rotations (Fig. 3b)

Matrices

$A, B, C, D, E, F, G, H, J, M, N, P, Q, R, S, T, X, Y, \Psi$ = 4×4 matrices
K, L = 4×3 matrices
$\Lambda, \Omega, \beta, \eta$ = 4×4 diagonal matrices
$e, f, g, l, x, y, z, \sigma_T$ = 1×4 column matrices

Indices

i = station
j = discontinuity station
m = mth discontinuity station (subscript on j)
N = last station
p = number of discontinuity stations
n = Fourier component

Introduction

NUMERICAL methods recently have been devised for the elastic, small deflection analysis of thin shells of revolution loaded axisymmetrically[1, 2]; the present paper extends these methods to the analysis of such shells subjected to arbitrary load distributions. The shell material is assumed to have two-dimensional elastic isotropy with respect to directions tangent to its surface, but Young's modulus is permitted to be variable (and discontinuous) through the thickness as well as in the meridional direction. For simplicity, Poisson's ratio is assumed constant. Thermal strain effects due to arbitrary temperature distributions are included in the analysis; however, any influence of temperature on Young's modulus must be averaged circumferentially in order to preserve the essential requirement of physical symmetry about the shell axis. Accordingly, the present analysis is inapplicable when the circumferential variation of temperature is sufficiently great to produce appreciable circumferential changes in Young's modulus.

The analysis is based on the general first-order linear shell theory of Sanders,[3] which has been assessed[4] as the "best" of the many competing thin-shell theories in the literature. All pertinent variables are expanded into Fourier series in the circumferential direction, and decoupled sets of ordinary differential equations thereby are obtained for the individual Fourier components of the independent variables sought. Finite-difference approximations to these differential equations then are solved by means of appropriate matrix techniques.

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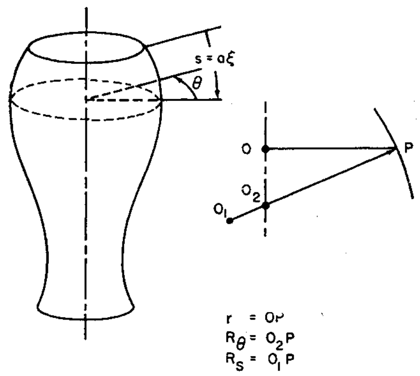


Fig. 1 Surface geometry and coordinates.

Surface Geometry and Coordinates

Material points in the shell can be specified by means of the orthogonal coordinates (s, θ, ζ), where (see Fig. 1) s is the meridional distance measured from a boundary along an axisymmetric reference surface, θ is the circumferential angle, and ζ is the normal, outward distance from the reference surface. In homogeneous shells, the middle surface always is used as the reference surface; but when, more generally, the Young's modulus E is variable, the reference surface is best chosen so that

$$\int \zeta E d\zeta = 0 \tag{1}$$

where the integration is through the thickness. (This choice, as will be seen later, simplifies the constitutive relations of elastic shells.) If the shape of the reference surface is given by $r(s)$, where r is the distance from the axis, the principal radii of curvature are

$$\begin{aligned} R_\theta &= r[1 - (dr/ds)^2]^{-1/2} \\ R_s &= -[1 - (dr/ds)^2]^{1/2}/(d^2r/ds^2) \end{aligned} \tag{2}$$

Introduce the nondimensional meridional coordinate $\xi = s/a$, where a is a reference length; then, with $\rho = r/a$, the nondimensional curvatures $\omega_\xi = a/R_s$ and $\omega_\theta = a/R_\theta$ can be found from the formulas

$$\omega_\theta = [1 - (\rho')^2]^{1/2}/\rho \tag{3}$$

$$\omega_\xi = -(\gamma' + \gamma^2)/\omega_\theta \tag{4}$$

where

$$\gamma = \rho'/\rho \tag{5}$$

In these equations, and henceforth, $(\ )' \equiv (d/d\xi)(\ )$. Finally, note the Codazzi identity

$$\omega_\theta' = \gamma(\omega_\xi - \omega_\theta) \tag{6}$$

and the relation

$$\rho''/\rho = -\omega_\xi\omega_\theta \tag{7}$$

Analytical Formulations

Forces, Moments, and Loads

The components of membrane force per unit length, transverse force per unit length, moment (about the reference surface) per unit length, and load per unit area (assumed to be applied at the reference surface) are as shown in Fig. 2. In the Sanders theory, the shearing forces $N_{\xi\theta}$ and $N_{\theta\xi}$, as well as the twisting moments $M_{\xi\theta}$ and $M_{\theta\xi}$, are not handled separately but are combined to provide the modified variables

$$\bar{N}_{\xi\theta} = \frac{1}{2}(N_{\xi\theta} + N_{\theta\xi}) + \frac{1}{4}\left(\frac{1}{R_\theta} - \frac{1}{R_\xi}\right)(M_{\xi\theta} - M_{\theta\xi}) \tag{8}$$

and

$$\bar{M}_{\xi\theta} = \tfrac{1}{2}(M_{\xi\theta} + M_{\theta\xi}) \tag{9}$$

With the elimination of the transverse forces Q_ξ and Q_θ, the equilibrium equation of the Sanders theory[3] can be written, for shells of revolution, as

$$a\left[\frac{\partial}{\partial\xi}(\rho N_\xi) + \frac{\partial}{\partial\theta}(\bar{N}_{\xi\theta}) - \rho' N_\theta\right] + \omega_\xi\left[\frac{\partial}{\partial\xi}(\rho M_\xi) + \frac{\partial}{\partial\theta}(\bar{M}_{\xi\theta}) - \rho' M_\theta\right] + \frac{1}{2}(\omega_\xi - \omega_\theta)\frac{\partial}{\partial\theta}(\bar{M}_{\xi\theta}) + a^2\rho q_\xi = 0 \tag{10a}$$

$$a\left[\frac{\partial}{\partial\theta}(N_\theta) + \frac{\partial}{\partial\xi}(\rho\bar{N}_{\xi\theta}) + \rho'\bar{N}_{\xi\theta}\right] + \omega_\theta\left[\frac{\partial}{\partial\theta}(M_\theta) + \frac{\partial}{\partial\xi}(\rho\bar{M}_{\xi\theta}) + \rho'\bar{M}_{\xi\theta}\right] + \frac{\rho}{2}\frac{\partial}{\partial\xi}[(\omega_\theta - \omega_\xi)\bar{M}_{\xi\theta}] + a^2\rho q_\theta = 0 \tag{10b}$$

$$\frac{\partial}{\partial\xi}\left[\frac{\partial}{\partial\xi}(\rho M_\xi) + \frac{\partial}{\partial\theta}(\bar{M}_{\xi\theta}) - \rho' M_\theta\right] + \frac{1}{\rho}\frac{\partial}{\partial\theta}\left[\frac{\partial}{\partial\theta}(M_\theta) + \frac{\partial}{\partial\xi}(\rho\bar{M}_{\xi\theta}) + \rho'\bar{M}_{\xi\theta}\right] - a\rho(\omega_\xi N_\xi + \omega_\theta N_\theta) + a^2\rho q = 0 \tag{10c}$$

These equations are exact.[4]

Displacements, Rotations, and Strains

The displacements and rotations of the reference surface (Fig. 3) are related by the equations

$$\begin{aligned} \Phi_\xi &= \frac{1}{a}\left[-\frac{\partial W}{\partial\xi} + \omega_\xi U_\xi\right] \\ \Phi_\theta &= \frac{1}{a}\left[-\frac{1}{\rho}\frac{\partial W}{\partial\theta} + \omega_\theta U_\theta\right] \end{aligned} \tag{11}$$

The membrane strains of the reference surface are given by

$$\begin{aligned} \epsilon_\xi &= \frac{1}{a}\left[\frac{\partial U_\xi}{\partial\xi} + \omega_\xi W\right] \\ \epsilon_\theta &= \frac{1}{a}\left[\frac{1}{\rho}\frac{\partial U_\theta}{\partial\theta} + \gamma U_\xi + \omega_\theta W\right] \\ \epsilon_{\xi\theta} &= \frac{1}{2a}\left[\frac{1}{\rho}\frac{\partial U_\xi}{\partial\theta} + \frac{\partial U_\theta}{\partial\xi} - \gamma U_\theta\right] \end{aligned} \tag{12}$$

where $\epsilon_{\xi\theta}$ is half the usual engineering shear strain.

Finally, the measures of bending distortion used in the Sanders theory are

$$\begin{aligned} \kappa_\xi &= \frac{1}{a}\frac{\partial\Phi_\xi}{\partial\xi} \\ \kappa_\theta &= \frac{1}{a}\left[\frac{1}{\rho}\frac{\partial\Phi_\theta}{\partial\theta} + \gamma\Phi_\xi\right] \\ \kappa_{\xi\theta} &= \frac{1}{2a}\left[\frac{1}{\rho}\frac{\partial\Phi_\xi}{\partial\theta} + \frac{\partial\Phi_\theta}{\partial\xi} - \gamma\Phi_\theta + \frac{1}{2a}(\omega_\xi - \omega_\theta)\left(\frac{1}{\rho}\frac{\partial U_\xi}{\partial\theta} - \frac{\partial U_\theta}{\partial\xi} - \gamma U_\theta\right)\right] \end{aligned} \tag{13}$$

Then, by the usual Kirchhoff hypothesis ("normals remain normal") and the neglect of terms of order ζ/R_s and ζ/R_θ relative to unity, the longitudinal, circumferential, and shear strains at a distance ζ from the reference surface are

$$\begin{array}{c} \epsilon_\xi + \zeta\kappa_\xi \\ \epsilon_\theta + \zeta\kappa_\theta \\ \epsilon_{\xi\theta} + \zeta\kappa_{\xi\theta} \end{array} \tag{14}$$

respectively.

Constitutive Relations

Neglecting, as usual, the effects of stresses normal to the shell permits the stress-strain-temperature relations to be written as

$$\begin{aligned} \epsilon_\xi + \zeta\kappa_\xi &= [(\sigma_\xi - \nu\sigma_\theta)/E] + \alpha T \\ \epsilon_\theta + \zeta\kappa_\theta &= [(\sigma_\theta - \nu\sigma_\xi)/E] + \alpha T \\ \epsilon_{\xi\theta} + \zeta\kappa_{\xi\theta} &= [(1+\nu)/E]\sigma_{\xi\theta} \end{aligned} \tag{15}$$

where the temperature change T may vary with ζ, as well as with ξ and θ. The Young's modulus E and the thermal expansion coefficient α will, however, be permitted to vary only with ξ and ζ. The (modified) forces and moments are approximated closely in the shell by the following integrals through the thickness:

$$\begin{array}{ll} N_\xi = \int \sigma_\xi d\zeta & M_\xi = \int \zeta\sigma_\xi d\zeta \\ N_\theta = \int \sigma_\theta d\zeta & M_\theta = \int \zeta\sigma_\theta d\zeta \\ \bar{N}_{\xi\theta} = \int \sigma_{\xi\theta} d\zeta & \bar{M}_{\xi\theta} = \int \zeta\sigma_{\xi\theta} d\zeta \end{array} \tag{16}$$

Then, with the use of the defining relation (1) for the reference surface, together with the assumption of constant Poisson's ratio, it is found from (14–16) that

$$\begin{aligned} \epsilon_\xi &= \frac{N_\xi - \nu N_\theta}{\int E d\zeta} + \frac{\int E\alpha T d\zeta}{\int E d\zeta} \\ \epsilon_\theta &= \frac{N_\theta - \nu N_\xi}{\int E d\zeta} + \frac{\int E\alpha T d\zeta}{\int E d\zeta} \\ \epsilon_{\xi\theta} &= \frac{(1+\nu)\bar{N}_{\xi\theta}}{\int E d\zeta} \end{aligned} \tag{17}$$

and

$$\begin{aligned} \kappa_\xi &= \frac{M_\xi - \nu M_\theta}{\int \zeta^2 E d\zeta} + \frac{\int \zeta E\alpha T d\zeta}{\int \zeta^2 E d\zeta} \\ \kappa_\theta &= \frac{M_\theta - \nu M_\xi}{\int \zeta^2 E d\zeta} + \frac{\int \zeta E\alpha T d\zeta}{\int \zeta^2 E d\zeta} \\ \kappa_{\xi\theta} &= \frac{(1+\nu)\bar{M}_{\xi\theta}}{\int \zeta^2 E d\zeta} \end{aligned} \tag{18}$$

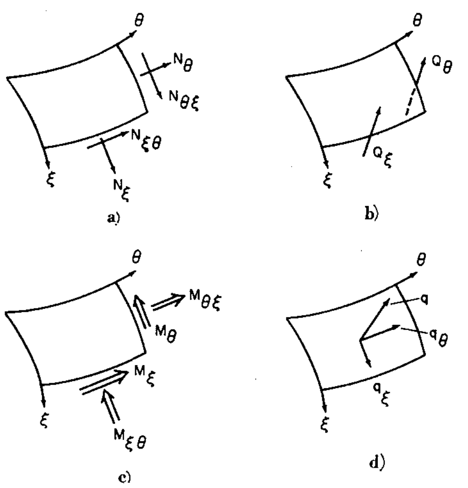


Fig. 2 Forces, moments, and loads; a) membrane forces per unit length, b) transverse forces per unit length, c) moments per unit length, d) loads per unit area.

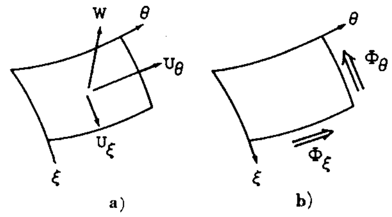


Fig. 3 a) Displacements; b) rotations.

The complete set of field equations for the 17 independent variables N_ξ, N_θ, $\bar{N}_{\xi\theta}$, M_ξ, M_θ, $\bar{M}_{\xi\theta}$, U_ξ, U_θ, W, Φ_ξ, Φ_θ, ϵ_ξ, ϵ_θ, $\epsilon_{\xi\theta}$, κ_ξ, κ_θ, $\kappa_{\xi\theta}$ now is given by the 17 equations (10–13, 17, and 18).

Fourier Expansions and Nondimensional Equations

The independent variables now will be expanded into Fourier series, with appropriate normalization to provide nondimensional Fourier coefficients of roughly comparable magnitudes for the different variables. Letting σ_0 be a reference stress level, E_0 a reference Young's modulus, and h_0 a reference thickness, solutions of the field equations will be sought in the following forms:

$$\begin{aligned} N_\xi &= \sigma_0 h_0 \sum_{n=0}^{\infty} t_\xi^{(n)} \cos n\theta \\ N_\theta &= \sigma_0 h_0 \sum_{n=0}^{\infty} t_\theta^{(n)} \cos n\theta \\ \bar{N}_{\xi\theta} &= \sigma_0 h_0 \sum_{n=1}^{\infty} t_{\xi\theta}^{(n)} \sin n\theta \end{aligned} \tag{19}$$

$$\begin{aligned} M_\xi &= \frac{\sigma_0 h_0^3}{a} \sum_{n=0}^{\infty} m_\xi^{(n)} \cos n\theta \\ M_\theta &= \frac{\sigma_0 h_0^3}{a} \sum_{n=0}^{\infty} m_\theta^{(n)} \cos n\theta \\ \bar{M}_{\xi\theta} &= \frac{\sigma_0 h_0^3}{a} \sum_{n=1}^{\infty} m_{\xi\theta}^{(n)} \sin n\theta \end{aligned} \tag{20}$$

$$\begin{aligned} U_\xi &= \frac{a\sigma_0}{E_0} \sum_{n=0}^{\infty} u_\xi^{(n)} \cos n\theta \\ U_\theta &= \frac{a\sigma_0}{E_0} \sum_{n=1}^{\infty} u_\theta^{(n)} \sin n\theta \\ W &= \frac{a\sigma_0}{E_0} \sum_{n=0}^{\infty} w^{(n)} \cos n\theta \end{aligned} \tag{21}$$

$$\begin{aligned} \Phi_\xi &= \frac{\sigma_0}{E_0} \sum_{n=0}^{\infty} \varphi_\xi^{(n)} \cos n\theta \\ \Phi_\theta &= \frac{\sigma_0}{E_0} \sum_{n=1}^{\infty} \varphi_\theta^{(n)} \sin n\theta \end{aligned} \tag{22}$$

$$\begin{aligned} \epsilon_\xi &= \frac{\sigma_0}{E_0} \sum_{n=0}^{\infty} e_\xi^{(n)} \cos n\theta \\ \epsilon_\theta &= \frac{\sigma_0}{E_0} \sum_{n=0}^{\infty} e_\theta^{(n)} \cos n\theta \\ \epsilon_{\xi\theta} &= \frac{\sigma_0}{E_0} \sum_{n=1}^{\infty} e_{\xi\theta}^{(n)} \sin n\theta \end{aligned} \tag{23}$$

$$\kappa_\xi = \frac{\sigma_0}{aE_0} \sum_{n=0}^{\infty} k_\xi^{(n)} \cos n\theta$$

$$\kappa_\theta = \frac{\sigma_0}{aE_0} \sum_{n=0}^{\infty} k_\theta^{(n)} \cos n\theta \tag{24}$$

$$\kappa_{\xi\theta} = \frac{\sigma_0}{aE_0} \sum_{n=1}^{\infty} k_{\xi\theta}^{(n)} \sin n\theta$$

These Fourier expansions are consistent with loadings of the forms

$$q = \frac{\sigma_0 h_0}{a} \sum_{n=0}^{\infty} p^{(n)}(\xi) \cos n\theta$$

$$q_\xi = \frac{\sigma_0 h_0}{a} \sum_{n=0}^{\infty} p_\xi^{(n)}(\xi) \cos n\theta \tag{25}$$

$$q_\theta = \frac{\sigma_0 h_0}{a} \sum_{n=1}^{\infty} p_\theta^{(n)}(\xi) \sin n\theta$$

and a temperature distribution

$$T = \sum_{n=0}^{\infty} T^{(n)}(\xi,\zeta) \cos n\theta \tag{26}$$

The various field equations now can be decoupled into separate sets for each Fourier index n; *for convenience, the superscript* (n) *on Fourier coefficients will be omitted in the equations that follow.* The equilibrium equations (10) lead to

$$\begin{aligned} &t_\xi' + \gamma(t_\xi - t_\theta) + (n/\rho)t_{\xi\theta} + \lambda^2\{\omega_\xi m_\xi' + \\ &\quad \gamma\omega_\xi(m_\xi - m_\theta) + (n/2\rho)(3\omega_\xi - \omega_\theta)m_{\xi\theta}\} + p_\xi = 0 \\ &t_{\xi\theta}' + 2\gamma t_{\xi\theta} - (n/\rho)t_\theta + \lambda^2\{-(n/\rho)\omega_0 m_\theta + \\ &\tfrac{1}{2}(3\omega_\theta - \omega_\xi)m_{\xi\theta}' + \tfrac{1}{2}[\gamma(3\omega_\theta + \omega_\xi) - \omega_\xi']m_{\xi\theta}\} + p_\theta = 0 \\ &-\omega_\xi t_\xi - \omega_\theta t_\theta + \lambda^2\{m_\xi'' + 2\gamma m_\xi' - \omega_\xi\omega_0 m_\xi + \\ &\quad [\omega_\xi\omega_\theta - (n^2/\rho^2)]m_\theta - \gamma m_\theta' + (2n/\rho)m_{\xi\theta}' + \\ &\qquad (2\gamma n/\rho)m_{\xi\theta}\} + p = 0 \end{aligned} \tag{27}$$

where $\lambda = h_0/a$, and use has been made of the geometrical identities (6) and (7). The relations (11–13) give

$$\varphi_\xi = -w' + \omega_\xi u_\xi \tag{28a}$$

$$\varphi_\theta = (n/\rho)w + \omega_\theta u_\theta \tag{28b}$$

$$\begin{aligned} e_\xi &= u_\xi' + \omega_\xi w \\ e_\theta &= (n/\rho)u_\theta + \gamma u_\xi + \omega_0 w \\ e_{\xi\theta} &= \tfrac{1}{2}[u_\theta' - \gamma u_\theta - (n/\rho)u_\xi] \end{aligned} \tag{29}$$

$$\begin{aligned} &k_\xi = \varphi_\xi' \qquad k_\theta = (n/\rho)\varphi_\theta + \gamma\varphi_\xi \\ &k_{\xi\theta} = \tfrac{1}{2}\{-(n/\rho)\varphi_\xi + \varphi_\theta' - \gamma\varphi_\theta + \\ &\qquad \tfrac{1}{2}(\omega_\theta - \omega_\xi)[(nu_\xi/\rho) + u_\theta' + \gamma u_\theta]\} \end{aligned} \tag{30}$$

and, finally, the constitutive relations (17) and (18), inverted to give forces and moments in terms of strains and bending distortions, lead to

$$\begin{aligned} t_\xi = b(e_\xi + \nu e_\theta) - t_T^{(n)} \qquad t_\theta = b(e_\theta + \nu e_\xi) - t_T^{(n)} \\ t_{\xi\theta} = b(1 - \nu)e_{\xi\theta} \end{aligned} \tag{31}$$

and

$$m_\xi = d(k_\xi + \nu k_\theta) - m_T^{(n)} \tag{32a}$$

$$m_\theta = d(k_\theta + \nu k_\xi) - m_T^{(n)} \tag{32b}$$

$$m_{\xi\theta} = d(1 - \nu)k_{\xi\theta} \tag{32c}$$

where

$$b = \frac{\int E d\zeta}{E_0 h_0(1 - \nu^2)} \tag{33}$$

$$d = \frac{\int \zeta^2 E d\zeta}{E_0 h_0^3(1 - \nu^2)} \tag{34}$$

$$t_T^{(n)} = \frac{\int E\alpha T^{(n)} d\zeta}{\sigma_0 h_0(1 - \nu)} \tag{35}$$

$$m_T^{(n)} = \frac{a\int \zeta E\alpha T^{(n)} d\zeta}{\sigma_0 h_0^3(1 - \nu)} \tag{36}$$

(Again, the superscript (n) on $t_T^{(n)}$ and $m_T^{(n)}$ will be omitted henceforth.)

For each n, the set of field equations for the 17 Fourier coefficients t_ξ, t_θ, $t_{\xi\theta}$, m_ξ, m_θ, $m_{\xi\theta}$, u_ξ, u_θ, w, φ_ξ, φ_θ, e_ξ, e_θ, $e_{\xi\theta}$, k_ξ, k_θ, $k_{\xi\theta}$ now is given by the 17 equations (27–32).

It may be remarked at this point that the Fourier expansions (25) and (26)—symmetrical about $\theta = 0$ for q, q_ξ, and T and antisymmetrical for q_θ—are not, of course, the most general that could exist. For full generality, these expansions should be augmented by the additional series

$$\bar{q} = \frac{\sigma_0 h_0}{a} \sum_{n=1}^{\infty} \bar{p}^{(n)}(\xi) \sin n\theta$$

$$\bar{q}_\xi = \frac{\sigma_0 h_0}{a} \sum_{n=1}^{\infty} \bar{p}_\xi^{(n)}(\xi) \sin n\theta$$

$$\bar{q}_\theta = \frac{\sigma_0 h_0}{a} \sum_{n=0}^{\infty} \bar{p}_\theta^{(n)}(\xi) \cos n\theta$$

$$\bar{T} = \sum_{n=1}^{\infty} \bar{T}^{(n)}(\xi,\zeta) \sin n\theta$$

But note, for example, that the contribution $\bar{p}^{(n)} \sin n\theta$ to the series for $\bar{q}$ may be written $\bar{p}^{(n)} \cos n[\theta - (\pi/n)]$, and so its effect on the shell is described by the equations already derived as long as an appropriate shift in the θ scale is incorporated into the interpretation of the results. A similar interpretation is applicable for all of the other barred Fourier coefficients, with the exception of $p_\theta^{(0)}$; but the loading associated with this term is one of pure torsion, which is best handled separately by membrane shell theory.

Reduction to Four Second-Order Differential Equations

The set of field equations obtained constitutes an eighth-order system that can be reduced, in a conventional fashion, to three equations in u_ξ, u_θ, and w. But a more attractive procedure is to derive four differential equations, each of second order, in the variables u_ξ, u_θ, w, and m_ξ. In so doing, it is necessary to eliminate m_θ by means of the relation

$$m_\theta = \nu m_\xi + d(1 - \nu^2)k_\theta - (1 - \nu)m_T \tag{37}$$

in order to prevent the ultimate appearance of derivatives of w of order higher than two. Then, substituting (37, 32c, and 31) into (27) and using (28–30) to eliminate the membrane strain and bending distortion gives three of the desired equations; the fourth equation is given by (32a), again with k_ξ and k_θ expressed in terms of the displacements. The resultant set then can be written as

$$\begin{aligned} &a_1u_\xi'' + a_2u_\xi' + a_3u_\xi + a_4u_\theta' + a_5u_\theta + a_6w' + \\ &\qquad a_7w + a_8m_\xi' + a_9m_\xi = C_1 \\ &a_{10}u_\xi' + a_{11}u_\xi + a_{12}u_\theta'' + a_{13}u_\theta' + a_{14}u_\theta + a_{15}w'' + \\ &\qquad a_{16}w' + a_{17}w + a_{18}m_\xi = C_2 \\ &a_{19}u_\xi' + a_{20}u_\xi + a_{21}u_\theta'' + a_{22}u_\theta' + a_{23}u_\theta + a_{24}w'' + \\ &\qquad a_{25}w' + a_{26}w + a_{27}m_\xi'' + a_{28}m_\xi' + a_{29}m_\xi = C_3 \\ &a_{30}u_\xi' + a_{31}u_\xi + a_{32}u_\theta + a_{33}w'' + a_{34}w' + \\ &\qquad a_{35}w + a_{36}m_\xi = C_4 \end{aligned} \tag{38}$$

where the a's and c's are given in Appendix A. These equations can be written in the matrix form

$$Ez'' + Fz' + Gz = e \tag{39}$$

where

$$z = \begin{bmatrix} u_\xi \\ u_\theta \\ w \\ m_\xi \end{bmatrix} \tag{40}$$

and

$$E = \begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_{12} & a_{15} & 0 \\ 0 & a_{21} & a_{24} & a_{27} \\ 0 & 0 & a_{33} & 0 \end{bmatrix} \qquad F = \begin{bmatrix} a_2 & a_4 & a_6 & a_8 \\ a_{10} & a_{13} & a_{16} & 0 \\ a_{19} & a_{22} & a_{25} & a_{28} \\ a_{30} & 0 & a_{34} & 0 \end{bmatrix}$$

$$G = \begin{bmatrix} a_3 & a_5 & a_7 & a_9 \\ a_{11} & a_{14} & a_{17} & a_{18} \\ a_{20} & a_{23} & a_{26} & a_{29} \\ a_{31} & a_{32} & a_{35} & a_{36} \end{bmatrix} \qquad e = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} \tag{41}$$

Boundary Conditions

In the Sanders theory, the expressions for virtual work per unit length at the boundaries $s = 0, \bar{s}$ are

$$\mp (N_\xi U_\xi + \hat{N}_{\xi\theta} U_\theta + \hat{Q}_\xi W + M_\xi \Phi_\xi) \tag{42}$$

where

$$\hat{N}_{\xi\theta} = \bar{N}_{\xi\theta} + [(3/2R_\theta) - (1/2R_\xi)]\,\bar{M}_{\xi\theta} \tag{43}$$

and

$$\hat{Q}_\xi = (1/a\rho)[(\partial/\partial\xi)(\rho M_\xi) + 2(\partial \bar{M}_{\xi\theta}/\partial\theta) - \rho' M_\theta] \tag{44}$$

are "effective" membrane and transverse shears, respectively, per unit length (see Fig. 4). This form of the virtual work indicates the kinds of boundary conditions that can be imposed; thus, either N_ξ or U_ξ may be prescribed, either $\hat{N}_{\xi\theta}$ or U_θ may be prescribed, and so on; or, more generally, N_ξ and U_ξ may be related through an elastic constraint against meridional displacement; and analogous constraints can link $\hat{N}_{\xi\theta}$ and U_θ, $\hat{Q}_\xi$ and W, and M_ξ and Φ_ξ. Letting

$$\hat{N}_{\xi\theta} = \sigma_0 h_0 \sum_{n=1}^{\infty} \hat{t}_{\xi\theta}^{(n)} \sin n\theta$$
$$\hat{Q}_\xi = \sigma_0 h_0 \sum_{n=0}^{\infty} \hat{f}_\xi^{(n)} \cos n\theta \tag{45}$$

gives (dropping superscripts)

$$\hat{t}_{\xi\theta} = t_{\xi\theta} + (\lambda^2/2)(3\omega_\theta - \omega_\xi) m_{\xi\theta}$$
$$\hat{f}_\xi = \lambda^2[m_\xi' + \gamma(m_\xi - m_\theta) + (2n/\rho) m_{\xi\theta}] \tag{46}$$

Then the boundary conditions just discussed always can be written (for the nth Fourier components) as

$$\Omega y + \Lambda z = l \tag{47}$$

where

$$y = \begin{bmatrix} t_\xi \\ \hat{t}_{\xi\theta} \\ \hat{f}_\xi \\ \varphi_\xi \end{bmatrix} \tag{48}$$

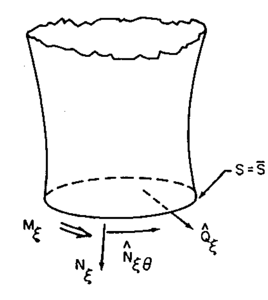


Fig. 4 Effective boundary forces and moment.

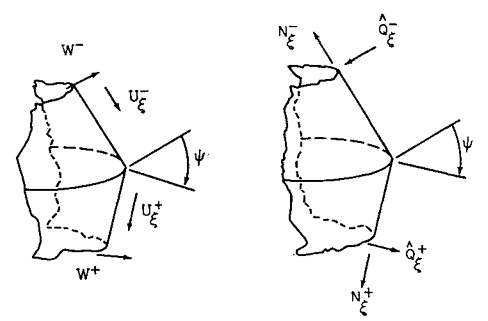


Fig. 5 Discontinuity conditions.

and where Ω and Λ are appropriate diagonal matrices, and l is a given column matrix. (For example, if u_ξ is given, the first diagonal element of Ω is zero, that of Λ is unity, and the first element of l is the prescribed value of u_ξ; if there is an elastic constraint on u_ξ, then the first diagonal element of Ω is unity, that of Λ is the appropriate constraint coefficient, and the first element of l vanishes.) But now it is desirable to express the boundary conditions entirely in terms of z; from Eqs. (28–32 and 37), it follows that

$$\begin{aligned} {}_{\xi} &= b_1 u_\xi' + b_2 u_\xi + b_3 u_\theta + b_4 w - t_T \\ {}_{\xi\theta} &= b_5 u_\xi + b_6 u_\theta' + b_7 u_\theta + b_8 w' + b_9 w \\ {}_{\xi} &= b_{10} u_\xi + b_{11} u_\theta' + b_{12} u_\theta + b_{13} w' + b_{14} w + b_{15} m_\xi' + \\ &\qquad b_{16} m_\xi + \lambda^2 \gamma (1 - \nu) m_T \end{aligned} \tag{49}$$

where the b's are given in Appendix A. These equations, together with (28a), then give

$$y = Hz' + Jz + f \tag{50}$$

where

$$H = \begin{bmatrix} b_1 & 0 & 0 & 0 \\ 0 & b_6 & b_8 & 0 \\ 0 & b_{11} & b_{13} & b_{15} \\ 0 & 0 & -1 & 0 \end{bmatrix} \qquad f = \begin{bmatrix} -t_T \\ 0 \\ \lambda^2\gamma(1-\nu) m_T \\ 0 \end{bmatrix}$$

$$J = \begin{bmatrix} b_2 & b_3 & b_4 & 0 \\ b_5 & b_7 & b_9 & 0 \\ b_{10} & b_{12} & b_{14} & b_{16} \\ \omega_\xi & 0 & 0 & 0 \end{bmatrix} \tag{51}$$

Hence, the boundary conditions (47) can be written as

$$\Omega H z' + (\Lambda + \Omega J) z = l - \Omega f \tag{52}$$

Discontinuity Conditions

The differential Eqs. (39) are not valid at points in the shell where discontinuities in geometry (and hence in the coefficients) occur; furthermore, z itself is ambiguous at a discontinuity in the inclination of the reference surface, where the directions of u_ξ and w change abruptly. Accordingly, special transition equations will be derived which relate z and its derivative on either side of a discontinuity. With plus and minus superscripts denoting values just beyond and ahead of a discontinuity, respectively, the conditions of geometrical compatibility are (see Fig. 5)

$$\begin{aligned} u_\xi^+ &= u_\xi^- \cos\psi - w^- \sin\psi \\ u_\theta^+ &= u_\theta^- \\ w^+ &= u_\xi^- \sin\psi + w^- \cos\psi \\ \varphi_\xi^+ &= \varphi_\xi^- \end{aligned} \tag{53}$$

and equilibrium requires that

$$\begin{aligned} t_\xi^+ &= t_\xi^- \cos\psi - \hat{f}_\xi^- \sin\psi \\ \hat{t}_{\xi\theta}^+ &= \hat{t}_{\xi\theta}^- \\ \hat{f}_\xi^+ &= t_\xi^- \sin\psi + \hat{f}_\xi^- \cos\psi \\ m_\xi^+ &= m_\xi^- \end{aligned} \tag{54}$$

(These last equations easily can be generalized to include the effects of externally applied circumferential line loads and moments.) The information in (53) and (54) is reproduced in the equations

$$y^+ = \Psi y^- \tag{55}$$

$$z^+ = \Psi z^- \tag{56}$$

where

$$\Psi = \begin{bmatrix} \cos\psi & 0 & -\sin\psi & 0 \\ 0 & 1 & 0 & 0 \\ \sin\psi & 0 & \cos\psi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{57}$$

Combining (55, 56, and 50) then provides the single equation relating $(z')^+$, $(z')^-$, and z^-:

$$H^+(z')^+ + (J^+\Psi - \Psi J^-)Z^- - \Psi H^-(z')^- = \Psi f^- - f^+ \tag{58}$$

where the plus and minus superscripts on H, J, and f mean that these matrices are to be calculated from (51) on the basis of shell properties just beyond and just ahead of the discontinuity, respectively. The differential equation (39), the boundary conditions (52), and the discontinuity conditions (58) now will be cast into a unified set of appropriate finite-difference equations.

Numerical Analysis

Finite-Difference Formulation

Suppose that p discontinuity locations $s_1, s_2, \ldots, s_p$ occur in the range $(0, \bar{s})$ of the shell; let the regions $(0, s_1)$, (s_1, s_2), ... $(s_p, \bar{s})$ be subdivided into V_1, V_2, ... V_{p+1} equal segments, respectively, and identify the endpoints of the segments by the index i, running from zero at $s = 0$ to $N(=\Sigma V)$ at $s = \bar{s}$. The increments in the nondimensional variable ξ are then

$$\begin{aligned} \Delta_1 &= s_1/aV_1 \\ \Delta_2 &= (s_2 - s_1)/aV_2 \\ &\vdots \\ \Delta_{p+1} &= (\bar{s} - s_p)/aV_{p+1} \end{aligned} \tag{59}$$

in the successive regions bounded by discontinuities. *Note that fictitious discontinuities may be inserted wherever a change in the increment size is considered desirable.* Finally, denote the discontinuity stations by $i = j_m$ $(m = 1, 2, \ldots p)$. The differential equations (39) will be written in finite difference form at all stations except $i = 0, j_m$ $(m = 1, 2, \ldots p)$, and N on the basis of the usual central difference formulas:

$$z_i'' = (z_{i+1} - 2z_i + z_{i-1})/\Delta^2 \tag{60}$$

$$z_i' = (z_{i+1} - z_{i-1})/2\Delta \tag{61}$$

where the Δ must, of course, be the one corresponding to the region associated with the station i. It should be noted that, when (60) and (61) are used at $i = j_m + 1$ (that is, at a station immediately *following* a discontinuity), z_{i-1}^+ must be used for z_{i-1}; similarly, when $i = j_m - 1$, z_{i+1}^- must be used for z_{i+1}.

The discontinuity equations (58) will be approximated at $i = j_n$ $(m = 1, 2, \ldots p)$ on the basis of the formulas

$$(z_j')^+ = (z_{j+1} - z_j^+)/\Delta^+ \tag{62}$$

$$(z_j')^- = (z_j^- - Z_{j-1})/\Delta^- \tag{63}$$

where, for simplicity, the subscript m on j has been omitted, and where Δ^+ and Δ^- are the intervals ahead and beyond station j_m, respectively; in fact, by (59), $\Delta^- = \Delta_m$, and $\Delta^+ = \Delta_{m+1}$.

Finally, the boundary conditions (52) will be written at $i = 0$ and $i = N$ with the help of

$$z_0' = (z_1 - z_0)/\Delta_1 \tag{64}$$

$$z_N' = (z_N - z_{N-1})/\Delta_p \tag{65}$$

The convention now will be adopted that whenever z_j is written without a qualifying superscript it means z_j^-; then, whenever z_j^+ appears it will be replaced by $\Psi_j z_j$, according to Eq. (56). Then, the results of writing the various difference equations just described can be stated compactly as the following set of algebraic equations for z_i $(i = 0, 1, 2, \ldots N)$:

$$\begin{aligned} A_0z_1 + B_0z_0 &= g_0 \\ A_iz_{i+1} + B_iz_i + C_iz_{i-1} &= g_i \qquad (i = 1, 2, \ldots N - 1) \\ B_Nz_N + C_Nz_{N-1} &= g_N \end{aligned} \tag{66}$$

Here

$$\begin{aligned} A_0 &= \Omega_0H_0/\Delta_1 \\ B_0 &= \Lambda_0 + \Omega_0[J_0 - (H_0/\Delta_1)] \\ g_0 &= l_0 - \Omega_0f_0 \end{aligned} \tag{67}$$

where the subscript zero refers, of course, to the conditions at $s = 0$. For $i \neq 0, j_m, j_m + 1, N$ $(m = 1, 2, \ldots p)$,

$$\begin{aligned} A_i &= (2E_i/\Delta) + F_i \\ B_i &= -(4E_i/\Delta) + 2\Delta G_i \\ C_i &= (2E_i/\Delta) - F_i \\ g_i &= 2\Delta e_i \end{aligned} \tag{68}$$

where the appropriate value for Δ is used. For $i = j_m + 1$, (68) applies, except that

$$C_{j+1} = \Psi_j[(2E_{j+1}/\Delta) - F_{j+1}] \tag{69}$$

For $i = j_m$,

$$\begin{aligned} A_j &= H_j^+/\Delta^+ \\ B_j &= [J_j^+ - (H_j^+/\Delta^+)]\Psi_j - \Psi_j[J_j^- + (H_j^-/\Delta^-)] \\ C_j &= \Psi_jH_j^-/\Delta^- \\ g_j &= \Psi_jf_j^- - f_j^+ \end{aligned} \tag{70}$$

Finally,

$$\begin{aligned} B_N &= \Lambda_N + \Omega_N[J_N + (H_N/\Delta p)] \\ C_N &= -(\Omega_NH_N/\Delta p) \\ g_N &= l_N - \Omega_Nf_N \end{aligned} \tag{71}$$

where N refers to the conditions at $s = \bar{s}$.

Matrix Solution of Difference Equations

The set of matrix equations (66) will be solved by essentially the same formal procedure used in Ref. 1 for the analogous equation for the case of axisymmetric loading of shells of revolution; this procedure actually is equivalent to solution by the method of Gaussian elimination used in Ref. 2 for the same axisymmetric loading problem. In its most primitive form, the Gaussian elimination technique would proceed as follows: the first of Eqs. (66) would be solved for z_0 in terms of z_1; this result would be substituted into the next equation, and z_1 would be found in terms of z_2 and so on; finally, the very last equation, together with the result for z_{N-1} in terms of z_N, would determine z_N, and then all of the z's would be calculated in reverse order. A minor modification of this method is, however, desirable (and sometimes essential) in the treatment of (66), for the matrix B_0 sometimes may be singular.‡ Accordingly, the solution is started

‡ This occurs, for example, in the case of a clamped edge, with $u_\xi = u_\theta = w = \varphi_\xi = 0$; then

$$l_0 = 0 \qquad \Omega_0 = \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 1 \end{bmatrix} \qquad \Lambda_0 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{bmatrix}$$

giving

$$B_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \omega_\xi & 0 & 1/\Delta & 0 \end{bmatrix}$$

which is singular.

by the simultaneous solution for z_0 and z_1 in terms of z_2 and then proceeds as just described. From

$$A_0 z_1 + B_0 z_0 = g_0$$
$$B_1 z_1 + C_1 z_0 = g_1 - A_1 z_2$$

it follows that

$$z_1 = -[B_0C_1^{-1}B_1 - A_0]^{-1}[B_0C_1^{-1}A_1z_2 - B_0C_1^{-1}g_1 + g_0] \quad (72)$$

Now write the general result for z_i in terms of z_{i+1} as

$$z_i = -P_i z_{i+1} + x_i \quad (73)$$
$$(i = 1, 2, \ldots N - 1)$$

Then, the substitution of $z_{i-1} = -P_{i-1}z_i + x_{i-1}$ into the general equation of (66) provides the results

$$P_i = [B_i - C_iP_{i-1}]^{-1}A_i$$
$$x_i = [B_i - C_iP_{i-1}]^{-1}[g_i - C_ix_{i-1}] \quad (74)$$
$$(i = 2,3, \ldots N - 1)$$

The recurrence relations (74), with the initial values from (72),

$$P_1 = [B_0C_1^{-1}B_1 - A_0]^{-1}B_0C_1^{-1}A_1$$
$$x_1 = [B_0C_1^{-1}B_1 - A_0]^{-1}[B_0C_1^{-1}g_1 - g_0] \quad (75)$$

then provide all the P's and x's up to P_{N-1} and x_{N-1}. Substitution of $z_{N-1} = -P_{N-1}z_N + x_{N-1}$ into the last of Eqs. (66) then gives

$$z_N = [B_N - C_NP_{N-1}]^{-1}[g_N - C_Nx_{N-1}] \quad (76)$$

and then $z_{N-1}, z_{N-2}, \ldots z_1$ can be found from (73). Finally, z_0 is given by

$$z_0 = C_1^{-1}[g_1 - A_1z_2 - B_1z_1] \quad (77)$$

Thus, the only matrix inversions involved in the solution for all the z's are of 4×4 matrices, and the process is suited very well for rapid machine computation.

The z_j obtained at a discontinuity station is, of course, really z_j^-; the value of z_j^+, at such a station easily is found as $z_j^+ = \Psi_j z_j^-$.

Calculation of Stresses

Once the z's have been calculated, the stresses at any point in the shell can be found. The stresses in the present solution are obtained from the expansions

$$\sigma_\xi = \sum_{n=0}^{\infty} \sigma_\xi^{(n)} \cos n\theta$$
$$\sigma_\theta = \sum_{n=0}^{\infty} \sigma_\theta^{(n)} \cos n\theta \quad (78)$$
$$\sigma_{\xi\theta} = \sum_{n=1}^{\infty} \sigma_{\xi\theta}^{(n)} \sin n\theta$$

Inverting the constitutive relations (15) and using (23, 24, and 26) gives

$$\sigma_\xi^{(n)} = \frac{E\sigma_0}{E_0(1-\nu^2)}\left[e_\xi^{(n)} + \nu e_\theta^{(n)} + \frac{\zeta}{a}(k_\xi^{(n)} + \nu k_\theta^{(n)})\right] - \frac{E\alpha T^{(n)}}{1-\nu}$$

$$\sigma_\theta^{(n)} = \frac{E\sigma_0}{E_0(1-\nu^2)}\left[e_\theta^{(n)} + \nu e_\xi^{(n)} + \frac{\zeta}{a}(k_\theta^{(n)} + \nu k_\theta^{(n)})\right] - \frac{E\alpha T^{(n)}}{1-\nu} \quad (79)$$

$$\sigma_{\xi\theta}^{(n)} = \frac{E\sigma_0}{E_\theta(1+\nu)}\left[e_{\xi\theta}^{(n)} + \frac{\zeta}{a}k_{\xi\theta}^{(n)}\right]$$

Note that E, α, and $T^{(n)}$ all may depend on ζ, the distance from the reference surface.

Using (32a, 32b, and 37) (and, again, casually dropping superscripts n) gives

$$k_\xi + \nu k_\theta = \frac{m_\xi + m_T}{d}$$

$$k_\theta + \nu k_\xi = \frac{m_\theta + m_T}{d} = \frac{\nu(m_\xi + m_T)}{d} + (1-\nu^2)k_\theta$$

which, when used in (79) together with the strain-rotation-displacement equations (28–30), leads to

$$\begin{bmatrix} \sigma_\xi^{(n)} \\ \sigma_\theta^{(n)} \\ \sigma_{\xi\theta}^{(n)} \end{bmatrix} = Kz' + Lz + \sigma_T \quad (80)$$

where

$$K = \frac{E\sigma_0}{E_0(1-\nu^2)}\begin{bmatrix} 1 & 0 & 0 & 0 \\ \nu & 0 & -\frac{\zeta\gamma(1-\nu^2)}{a} & 0 \\ 0 & \frac{1-\nu}{2}\left[1 + \frac{\zeta}{2a}(3\omega_\theta - \omega_\xi)\right] & \frac{\zeta}{a}(1-\nu)\frac{n}{\rho} & 0 \end{bmatrix} \quad (81)$$

$$L = \frac{E\sigma_0}{E_0(1-\nu^2)}\begin{bmatrix} \nu\gamma & \nu n/\rho & \omega_\xi + \nu\omega_\theta & \zeta/ad \\ \gamma\left[1 + \frac{(1-\nu^2)\zeta\omega_\xi}{a}\right] & \frac{n}{\rho}\left[1 + \frac{(1-\nu^2)\zeta\omega_\theta}{a}\right] & \omega_\theta + \nu\omega_\xi + \frac{(1-\nu^2)\zeta n^2}{a\rho^2} & \frac{\nu\zeta}{ad} \\ \left(\frac{1-\nu}{2}\right)\left(\frac{n}{\rho}\right)\left[-1 + \frac{(\omega_\theta - 3\omega_\xi)\zeta}{2a}\right] & \gamma\left(\frac{1-\nu}{2}\right)\left[-1 + \frac{(\omega_\xi - 3\omega_\theta)\zeta}{2a}\right] & -\frac{(1-\nu)n\gamma\zeta}{a\rho} & 0 \end{bmatrix} \quad (82)$$

$$\sigma_T = \begin{bmatrix} \frac{E\sigma_0\zeta m_T}{E_0(1-\nu^2)ad} - \frac{E\alpha T^{(n)}}{1-\nu} \\ \frac{\nu E\sigma_0\zeta m_T}{E_0(1-\nu^2)ad} - \frac{E\alpha T^{(n)}}{1-\nu} \\ 0 \end{bmatrix} \quad (83)$$

For numerical calculation, the use of (61) at "ordinary" stations, (62) and (63) at discontinuity stations, and (64) and (65) at the boundaries is recommended for the evaluation of z' in Eq. (80).

Remark Concerning the Reference Surface

A substantial simplification in setting up the numerical analysis for computation may result from the observation that, in the spirit of thin-shell theory, errors of the order of the thickness in the specification of the reference surface can be tolerated in the formulation of the equation of equilibrium. It is recommended accordingly that the key geometric function $r(s)$ be stated with respect to a surface chosen simply according to convenience anywhere in the shell wall. In other words, the condition (1) need not be imposed insofar as calculations of the various geometrical parameters ρ, ω_θ, ω_ξ, and γ are concerned. Of course, if (1) can be satisfied easily in these calculations, there is no harm in doing so; but when, for example, the same shell is to be analyzed for several different temperature conditions with different resultant variations of Young's modulus, it is *not* recommended that new reference surfaces and new variations of ρ, ω_θ, etc., be calculated for each case. On the other hand, it *is* essential that the rigorous location of the reference surface enter into Eqs. (34) and (36) for the nondimensional bending stiffness d and the thermal moment m_T. Similarly, the correct value of ζ as measured from the true reference surface must be used in Eqs. (80–83) for the stresses.

Brief Discussion of Computation Program

There is little point in presenting, in all its ramifications, the machine program that has been developed for executing the numerical analysis. It may be helpful, however, to summarize very briefly the sequence of computing steps involved:

1) From the given geometrical specification $r(s)$ of the assumed (see the foregoing) reference surface and the locations s_1, s_2, ... s_p of discontinuity stations, subdivide the shell, identify stations $i = 0, 1, 2, \ldots N$, and calculate ρ, ω_θ, ω_ξ, ω_ξ', and γ at these stations, using differentiation formulas similar to Eqs. (61–65) (including the computation of separate values on either side of discontinuity stations).

2) From knowledge of the variation of E through the thickness at each station (and on either side of discontinuity stations), find the rigorous location of the reference surface according to Eq. (1), and then calculate b, d, b', and d' at each station.

3) For a particular n, tabulate $\rho^{(n)}$, $p_\xi^{(n)}$, $p_\theta^{(n)}$, $T^{(n)}(\zeta)$ at each station; calculate $n_T^{(n)}$, $m_T^{(n)}$ at each station.

4) For the assumed n, use the boundary conditions at $i = 0$ to calculate p_i, x_i ($i = 1, 2, \ldots N - 1$), using the a's and b's in Appendix A to compute, at each station *as needed*, the matrices A, B, C, and g entering into the calculations. Then use these P's and x's and the boundary conditions at $i = N$ to get z at each station.

5) For the assumed n, calculate Fourier coefficients for the stresses σ_ξ, σ_θ, $\sigma_{\xi\theta}$ where desired.

6) Repeat steps 3–5 for as many n's as needed to insure adequate convergence; finally, use appropriate Fourier combinations of the separate results to obtain displacement and stresses.

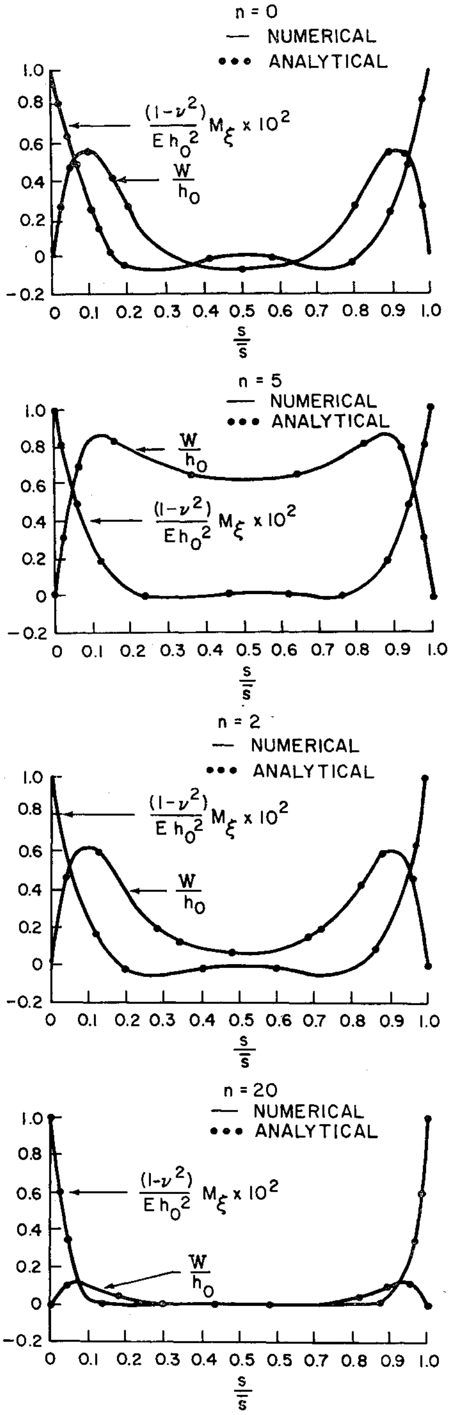


Fig. 6 Comparison of present solution with analytical solution for a cylinder with $U = V = W = 0$ and $(1 - \nu^2)M\xi/(Eh_0^2) = 10^{-2} \cos n\theta$ at the ends.

Check Calculation

A preliminary check of the calculation procedure has been made for the case of a uniform circular cylinder of thickness n_0 subjected to end moments $M\xi = 10^{-2}[Eh_0^2/(1 - \nu^2)] \cos n\theta$ with $U\xi = U_\theta = W = 0$ at the boundaries. An analytical solution of this problem on the basis of the Sanders theory was obtained for comparison. The calculations were made for a radius-to-thickness ratio $a/h_0 = 50$, a length-to-radius ratio $\bar{s}/a = 1$, and Poisson's ratio $\nu = 0.3$; 300 intervals were used over the length of the cylinder. The results for W/h_0 and $[M_\xi(1 - \nu^2)]/Eh_0^2$ at $\theta = 0$ are shown in Fig. 6 for $n = 0$, 2, 5, and 20, together with some analytically derived results. The agreement is excellent, despite the boundary layer character of the results for the higher values of n.

Supplementary Remarks

Singular Points

If the shell has a pole (i.e., $r = 0$), coefficients in the governing differential equations become singular. A simple-minded way to handle this situation is to choose the boundary $s = 0$ *not* at the pole, but a very short distance away, and then impose the boundary conditions at $s = 0$ as $u_\xi = u_\theta = \bar{f}_\xi = \varphi_\xi = 0$ for $n = 0$, $t_\xi = \hat{t}_{\xi\theta} = w = m_\xi = 0$ for $n = 1$ (assuming no concentrated forces or moments at the pole), amd $u_\xi = u_\theta = w = m_\xi = 0$ for all other n. This probably is not the most accurate or elegant procedure, and alternatives merit study.

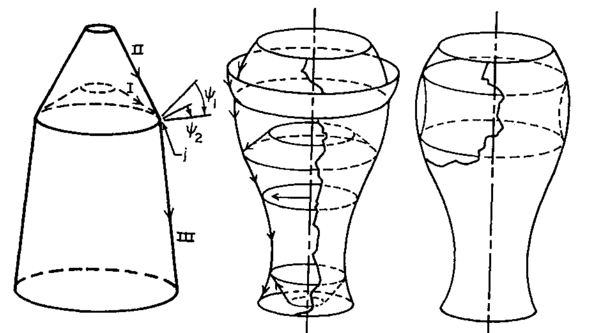
Branching Shells

It has been assumed tacitly all along that the shell under consideration has no more than two boundaries; a multiple-branch shell such as shown in Fig. 7a may be analyzed, however, by applying appropriate transition conditions at the branch point.

Define separate families of auxiliary matrices P^{I}, P^{II}, P^{III}, x^{I}, x^{II}, and x^{III} with the properties

$$z_i^{\mathrm{I}} = -P_i^{\mathrm{I}} z_{i+1}^{\mathrm{I}} + x_i^{\mathrm{I}}$$
$$z_i^{\mathrm{II}} = -P_i^{\mathrm{II}} z_{i+1}^{\mathrm{II}} + x_i^{\mathrm{II}}$$
$$z_i^{\mathrm{III}} = -P_i^{\mathrm{III}} z_{i+1}^{\mathrm{III}} + x_i^{\mathrm{III}}$$

where the superscripts refer to the separate branches shown in Fig. 7a. It is possible to start the calculations of $P^{\mathrm{I}}, x^{\mathrm{I}}$ and p^{II}, x^{II} at the boundaries of branches I and II and then leap across the juncture j to the calculation of $P^{\mathrm{III}}, x^{\mathrm{III}}$. The reverse sweep for the calculation of the z's then would start at the boundary of branch III and, at the juncture j, continue independently along the branches I and II back to their respective boundaries. The details of this procedure are given in Appendix B. This method can be extended readily to handle a multiplicity of branches as in Fig. 7b; it will not, however, be applicable to closed loops (Fig. 7c), which must be treated separately by traditional cut-and-fit methods of indeterminate structural analysis.



a) b) c)

Fig. 7 Branched shells.

Appendix A: Formulas for Coefficients

The coefficients $a_1, a_2 \ldots a_{36}$ in Eq. (38) are as follows:

$$a_1 = b \qquad a_2 = \gamma b + b'$$

$$a_3 = \nu b'\gamma - \nu b\omega_\xi\omega_\theta - b\gamma^2 - \frac{(1-\nu)bn^2}{2\rho^2} - \lambda^2 d(1-\nu)\left[(1+\nu)\gamma^2\omega_\xi^2 + \frac{(3\omega_\xi - \omega_\theta)^2 n^2}{8\rho^2}\right]$$

$$a_4 = \frac{(1+\nu)bn}{2\rho} + \frac{\lambda^2 dn(1-\nu)}{8\rho}(3\omega_\xi - \omega_\theta)(3\omega_\theta - \omega_\xi)$$

$$a_5 = \frac{\nu nb'}{\rho} - \left(\frac{3-\nu}{2\rho}\right)(\gamma bn) - \frac{\lambda^2 d(1-\nu)\gamma n}{\rho}\left[\frac{(3\omega_\xi - \omega_\theta)(3\omega_\theta - \omega_\xi)}{8} + (1+\nu)\omega_\xi\omega_\theta\right]$$

$$a_6 = b(\omega_\xi + \nu\omega_\theta) + \lambda^2 d(1-\nu)[(1+\nu)\gamma^2\omega_\xi + (n^2/2\rho^2)(3\omega_\xi - \omega_\theta)]$$

$$a_7 = b[\omega_\xi' + \gamma(\omega_\xi - \omega_\theta)] + b'(\omega_\xi + \nu\omega_\theta) - \frac{\lambda^2 d(1-\nu)\gamma n^2}{\rho^2}\left[\frac{3\omega_\xi - \omega_\theta}{2} + (1+\nu)\omega_\xi\right]$$

$$a_8 = \lambda^2\omega_\xi \qquad a_9 = \lambda^2(1-\nu)\gamma\omega_\xi \qquad a_{10} = -a_4$$

$$a_{11} = -\frac{b\gamma n}{2\rho}(3-\nu) - \frac{(1-\nu)nb'}{2\rho} + \frac{\lambda^2 d(1-\nu)n}{\rho} \times \left[-(1+\nu)\gamma\omega_\xi\omega_\theta + \frac{\gamma}{8}(6\omega_\xi\omega_\theta - 7\omega_\xi^2 - 3\omega_\theta^2) - \frac{\omega_\xi'}{4}(5\omega_\theta - 3\omega_\xi)\right] - \frac{\lambda^2 d'(1-\nu)n}{8\rho}(3\omega_\xi - \omega_\theta)(3\omega_\theta - \omega_\xi)$$

$$a_{12} = \frac{b(1-\nu)}{2} + \frac{\lambda^2 d(1-\nu)(3\omega_\theta - \omega_\xi)^2}{8}$$

$$a_{13} = \left(\frac{1-\nu}{2}\right)(\gamma b + b') - \frac{\lambda^2 d(1-\nu)}{8}(3\omega_\theta - \omega_\xi) \times [2\omega_\xi' - \gamma(5\omega_\xi - 3\omega_\theta)] + \frac{\lambda^2 d'(1-\nu)}{8}(3\omega_\theta - \omega_\xi)^2$$

$$a_{14} = -\gamma a_{13} + \left(\frac{1-\nu}{2}\right)b\omega_\xi\omega_\theta - \frac{bn^2}{\rho^2} - \lambda^2 d(1-\nu)\left[\frac{(1+\nu)\omega_\theta^2 n^2}{\rho^2} - \frac{\omega_\xi\omega_\theta}{8}(3\omega_\theta - \omega_\xi)^2\right]$$

$$a_{15} = \frac{\lambda^2 d(1-\nu)(3\omega_\theta - \omega_\xi)n}{2\rho}$$

$$a_{16} = \frac{\lambda^2 d(1-\nu)n}{2\rho}\left[2(1+\nu)\gamma\omega_\theta - \omega_\xi' + 3\gamma(\omega_\xi - \omega_\theta)\right] + \frac{\lambda^2 d'(1-\nu)(3\omega_\theta - \omega_\xi)n}{2\rho}$$

$$a_{17} = -\frac{bn(\omega_\theta + \nu\omega_\xi)}{\rho} + \frac{\lambda^2 dn(1-\nu)}{2\rho} \times \left[\gamma\omega_\xi' - 2\gamma^2\omega_\xi - \frac{2(1+\nu)\omega_\theta n^2}{\rho^2} + (3\omega_\theta - \omega_\xi)(\gamma^2 + \omega_\xi\omega_\theta)\right] - \frac{\lambda^2 d'n(1-\nu)\gamma}{2\rho}(3\omega_\theta - \omega_\xi)$$

$$a_{18} = -(\nu\lambda^2\omega_\theta n/\rho) \qquad a_{19} = -a_6$$

$$a_{20} = -b\gamma(\omega_\theta + \nu\omega_\xi) + \lambda^2 d(1-\nu)[\gamma(1+\nu)(-\gamma\omega_\xi' + \gamma^2\omega_\xi - (n^2\omega_\xi/\rho^2) + 2\omega_\xi^2\omega_\theta) + (n^2/2\rho^2)(\gamma\omega_\xi - \gamma\omega_\theta - 3\omega_\xi')] - \lambda^2 d'(1-\nu)[(1+\nu)\gamma^2\omega_\xi + (n^2/2\rho^2)(3\omega_\xi - \omega_\theta)]$$

$$a_{21} = a_{15}$$

$$a_{22} = \frac{\lambda^2 d(1-\nu)n}{2\rho}[3\gamma\omega_\xi - \gamma\omega_\theta(5+2\nu) - \omega_\xi'] + \frac{\lambda^2 d'(1-\nu)n}{2\rho}(3\omega_\theta - \omega_\xi)$$

$$a_{23} = -\frac{bn(\omega_\theta + \nu\omega_\xi)}{\rho} + \frac{\lambda^2 d(1-\nu)n}{2\rho} \times \left[2(1+\nu)\left(\omega_\xi\omega_\theta^2 - \gamma^2\omega_\xi + 2\gamma^2\omega_\theta - \frac{n^2\omega_\theta}{\rho^2}\right) + \gamma\omega_\xi' + 3\gamma^2(\omega_\theta - \omega_\xi) + \omega_\xi\omega_\theta(3\omega_\theta - \omega_\xi)\right] - \frac{\lambda^2 d'(1-\nu)n}{2\rho}[2(1+\nu)\gamma\omega_\theta + \gamma(3\omega_\theta - \omega_\xi)]$$

$$a_{24} = \lambda^2 d(1-\nu)[(2n^2/\rho^2) + (1+\nu)\gamma^2]$$

$$a_{25} = -\lambda^2 d(1-\nu)[(1+\nu)(2\gamma\omega_\xi\omega_\theta + \gamma^3) + (2\gamma n^2/\rho^2)] + \lambda^2 d'(1-\nu)[(1+\nu)\gamma^2 + (2n^2/\rho^2)]$$

$$a_{26} = -b(\omega_\xi^2 + 2\nu\omega_\xi\omega_\theta + \omega_\theta^2) + \frac{\lambda^2 d(1-\nu)n^2}{\rho^2}\left[(1+\nu)\left(\omega_\xi\omega_\theta - \frac{n^2}{\rho^2} + 2\gamma^2\right) + 2(\gamma^2 + \omega_\xi\omega_\theta)\right] - \frac{\lambda^2 d'(1-\nu)n^2}{\rho^2}(3+\nu)\gamma$$

$$a_{27} = \lambda^2 \qquad a_{28} = \lambda^2\gamma(2-\nu)$$

$$a_{29} = -\lambda^2[(1-\nu)\omega_\xi\omega_\theta + (\nu n^2/\rho^2)] \qquad a_{30} = d\omega_\xi$$

$$a_{31} = d(\omega_\xi' + \nu\gamma\omega_\xi) \qquad a_{32} = d\nu n\omega_\theta/\rho$$

$$a_{33} = -d \qquad a_{34} = -d\nu\gamma$$

$$a_{35} = d\nu n^2/\rho^2 \qquad a_{36} = -1$$

The c's are

$$c_1 = -p_\xi + t_T' - \lambda^2(1-\nu)\gamma\omega_\xi m_T$$

$$c_2 = -p_\theta - (n/\rho)t_T - \lambda^2(1-\nu)(n/\rho)\omega_\theta m_T$$

$$c_3 = -p - (\omega_\xi + \omega_\theta)t_T - \lambda^2(1-\nu)\gamma m_T' + \lambda^2(1-\nu)[\omega_\xi\omega_\theta - (n^2/\rho^2)]m_T$$

$$c_4 = m_T$$

F nally, the b's in Eq. (49) are

$$b_1 = b \qquad b_2 = \nu\gamma b$$

$$b_3 = \nu n b/\rho \qquad b_4 = b(\omega_\xi + \nu\omega_\theta)$$

$$b_5 = -\frac{b(1-\nu)n}{2\rho} - \frac{d\lambda^2(1-\nu)n}{8\rho}(3\omega_\xi - \omega_\theta)(3\omega_\theta - \omega_\xi)$$

$$b_6 = \frac{b(1-\nu)}{2} + \frac{\lambda^2 d(1-\nu)}{8}(3\omega_\theta - \omega_\xi)^2$$

$$b_7 = -\gamma b_6$$

$$b_8 = \frac{\lambda^2 d(1-\nu)n}{2\rho}(3\omega_\theta - \omega_\xi)$$

$$b_9 = -\gamma b_8$$

$$b_{10} = -\lambda^2 d(1-\nu)[(1+\nu)\gamma^2\omega_\xi + (n^2/2\rho^2)(3\omega_\xi - \omega_\theta)]$$

$$b_{11} = \frac{\lambda^2 d(1-\nu)n}{2\rho}(3\omega_\theta - \omega_\xi)$$

$$b_{12} = -\frac{\lambda^2 d(1-\nu)\gamma n}{2\rho}[3\omega_\theta - \omega_\xi + 2(1+\nu)\omega_\theta]$$

$$b_{13} = \lambda^2 d(1-\nu)[(2n^2/\rho^2) + (1+\nu)\gamma^2]$$

$$b_{14} = -\lambda^2 d(1-\nu)(3+\nu)(\gamma n^2/\rho^2)$$

$$b_{15} = \lambda^2 \qquad b_{16} = \lambda^2(1-\nu)\gamma$$

Appendix B: Branch Point Transition Relations

Recalling the definition (48) for the column matrix y and introducing the diagonal matrices

$$\beta = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{bmatrix} \qquad \eta = \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 1 \end{bmatrix}$$

permits the conditions of continuity (53) at the juncture in Fig. 7a to be written as

$$\beta z^{\mathrm{III}} + \eta y^{\mathrm{III}} = \beta\Psi^{\mathrm{II}}y^{\mathrm{II}} + \eta y^{\mathrm{II}} = \beta\Psi^{\mathrm{I}}z^{\mathrm{I}} + \eta y^{\mathrm{I}} \quad \text{(B1)}$$

where Ψ^{I} and Ψ^{II} are defined in terms of ψ_1 and ψ_2, respectively, according to Eq. (57). Similarly, the equilibrium conditions (54), generalized to the three-branch case, are

$$\beta y^{\mathrm{III}} + \eta z^{\mathrm{III}} = \beta\Psi^{\mathrm{II}}y^{\mathrm{II}} + \eta z^{\mathrm{II}} + \beta\Psi^{\mathrm{I}}y^{\mathrm{I}} + \eta z^{\mathrm{I}} \quad \text{(B2)}$$

Introducing Eq. (50) into (B1) and (B2) (and noting that $\eta f = 0$, $\beta f = f$) gives

$$\eta H^{\mathrm{III}}(z^{\mathrm{III}})' + (\beta + \eta J^{\mathrm{III}})z^{\mathrm{III}} = \eta H^{\mathrm{II}}(z^{\mathrm{II}})' + (\beta\Psi^{\mathrm{II}} + \eta J^{\mathrm{II}})z^{\mathrm{II}} = \eta H^{\mathrm{I}}(z^{\mathrm{I}})' + (\beta\Psi^{\mathrm{I}} + \eta J^{\mathrm{I}})z^{\mathrm{I}} \quad \text{(B3)}$$

$$\beta H^{\mathrm{III}}(z^{\mathrm{III}})' + (\eta + \beta J^{\mathrm{III}})z^{\mathrm{III}} = \beta\Psi^{\mathrm{II}}H^{\mathrm{II}}(z^{\mathrm{II}})' + (\eta + \beta\Psi^{\mathrm{II}}J^{\mathrm{II}})z^{\mathrm{II}} + \beta\Psi^{\mathrm{I}}H^{\mathrm{I}}(z^{\mathrm{I}})' + (\eta + \beta\Psi^{\mathrm{I}}J^{\mathrm{I}})z^{\mathrm{I}} + \Psi^{\mathrm{II}}f^{\mathrm{II}} + \Psi^{\mathrm{I}}f^{\mathrm{I}} - f^{\mathrm{III}} \quad \text{(B4)}$$

With the use of formulas (62) in branch III and (63) in branches I and II, Eqs. (B3) and (B4) can be cast into the difference forms:

$$Qz_{j+1}^{\mathrm{III}} + Rz_j^{\mathrm{III}} = S^{\mathrm{II}}z_j^{\mathrm{II}} + T^{\mathrm{II}}z_{j-1}^{\mathrm{II}} = S^{\mathrm{I}}z_j^{\mathrm{I}} + T^{\mathrm{I}}z_{j-1}^{\mathrm{I}} \quad \text{(B5)}$$

$$Mz_{j+1}^{\mathrm{III}} + Nz_j^{\mathrm{III}} = X^{\mathrm{II}}z_j^{\mathrm{II}} + Y^{\mathrm{II}}z_{j-1}^{\mathrm{II}} + X^{\mathrm{I}}z_j^{\mathrm{I}} + Y^{\mathrm{I}}z_{j-1}^{\mathrm{I}} + \Psi^{\mathrm{II}}f^{\mathrm{II}} + \Psi^{\mathrm{I}}f^{\mathrm{I}} - f^{\mathrm{III}} \quad \text{(B6)}$$

where the matrices Ω, R, etc., are derivable in obvious fashion.§

Using the basic Gaussian elimination formulas

$$z_{j-1}^{\mathrm{I}} = -P_{j-1}^{\mathrm{I}}z_j^{\mathrm{I}} + x_{j-1}^{\mathrm{I}}$$
$$z_{j-1}^{\mathrm{II}} = -P_{j-1}^{\mathrm{II}}z_j^{\mathrm{II}} + x_{j-1}^{\mathrm{II}}$$

in (B5) and (B6) gives

$$[S^{\mathrm{I}} - T^{\mathrm{I}}P_{j-1}^{\mathrm{I}}]z_j^{\mathrm{I}} + T^{\mathrm{I}}x_{j-1}^{\mathrm{I}} = [S^{\mathrm{II}} - T^{\mathrm{II}}P_{j-1}^{\mathrm{II}}]z_j^{\mathrm{II}} + T^{\mathrm{II}}x_{j-1}^{\mathrm{II}} = Qz_{j+1}^{\mathrm{III}} + Rz_j^{\mathrm{III}} \quad \text{(B7)}$$

$$[X^{\mathrm{I}} - Y^{\mathrm{I}}P_{j-1}^{\mathrm{I}}]z_j^{\mathrm{I}} + Y^{\mathrm{I}}x_{j-1}^{\mathrm{I}} + [X^{\mathrm{II}} - Y^{\mathrm{II}}P_{j-1}^{\mathrm{II}}]z_j^{\mathrm{II}} + Y^{\mathrm{II}}x_{j-1}^{\mathrm{II}} + \Psi^{\mathrm{II}}f^{\mathrm{II}} + \Psi^{\mathrm{I}}f^{\mathrm{I}} - f^{\mathrm{III}} = Mz_{j+1}^{\mathrm{III}} + Nz_j^{\mathrm{III}} \quad \text{(B8)}$$

Now eliminate z_j^{I} and z_j^{II} from (B8) by means of (B7):

$$[X^{\mathrm{I}} - Y^{\mathrm{I}}P_{j-1}^{\mathrm{I}}][S^{\mathrm{I}} - T^{\mathrm{I}}P_{j-1}^{\mathrm{I}}]^{-1}[Qz_{j+1}^{\mathrm{III}} + Rz_j^{\mathrm{III}} - T^{\mathrm{I}}x_{j-1}^{\mathrm{I}}] + [X^{\mathrm{II}} - Y^{\mathrm{II}}P_{j-1}^{\mathrm{II}}][S^{\mathrm{II}} - T^{\mathrm{II}}P_{j-1}^{\mathrm{II}}]^{-1}[Qz_{j+1}^{\mathrm{III}} + Rz_j^{\mathrm{III}} - T^{\mathrm{II}}x_{j-1}^{\mathrm{II}}] + Y^{\mathrm{I}}x_{j-1}^{\mathrm{I}} + Y^{\mathrm{II}}x_{j-1}^{\mathrm{II}} + \Psi^{\mathrm{II}}f^{\mathrm{II}} + \Psi^{\mathrm{I}}f^{\mathrm{I}} - f^{\mathrm{III}} = Mz_{j+1} + Nz_j^{\mathrm{III}} \quad \text{(B9)}$$

Hence, with

$$z_j^{\mathrm{III}} = -P_j^{\mathrm{III}}z_{j+1}^{\mathrm{III}} + x_j^{\mathrm{III}}$$

the desired expressions for P_j^{III} and x_j^{III} are, from (B9),

$$P_j^{\mathrm{III}} = [N - (D^{\mathrm{I}} + D^{\mathrm{II}})R]^{-1}[M - (D^{\mathrm{I}} + D^{\mathrm{II}})Q]$$
$$x_j^{\mathrm{III}} = [N - (D^{\mathrm{I}} + D^{\mathrm{II}})R]^{-1}[(Y^{\mathrm{I}} - D^{\mathrm{I}}T^{\mathrm{I}})x_{j-1}^{\mathrm{I}} + (Y^{\mathrm{II}} - D^{\mathrm{II}}T^{\mathrm{II}})x_{j-1}^{\mathrm{II}} + \Psi^{\mathrm{II}}f^{\mathrm{II}} + \Psi^{\mathrm{I}}f^{\mathrm{I}} - f^{\mathrm{III}}] \quad \text{(B10)}$$

where

$$D^{\mathrm{I}} = (X^{\mathrm{I}} - Y^{\mathrm{I}}P_{j-1}^{\mathrm{I}})(S^{\mathrm{I}} - T^{\mathrm{I}}P_{j-1}^{\mathrm{I}})^{-1}$$
$$D^{\mathrm{II}} = (X^{\mathrm{II}} - Y^{\mathrm{II}}P_{j-1}^{\mathrm{II}})(S^{\mathrm{II}} - T^{\mathrm{II}}P_{j-1}^{\mathrm{II}})^{-1}$$

Thus, from a knowledge of P_{j-1}^{I}, P_{j-1}^{II}, x_{j-1}^{I}, and x_{j-1}^{II}, the calculation can proceed directly to the determination of P_j^{III} and x_j^{III} and then to the boundary of branch III in the standard fashion.

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§ Note that R has exactly the same terms as the matrix B_0 discussed in the previous footnote and so is singular.